Chapter 5

Riemannian Manifolds

In this chapter we want to introduce the notion of an "intrinsic geometry" without making reference to an ambient space $\mathbb{R}^{n+1}$, not only locally, but also as a global notion. This continues the considerations of Chapter 4. The most important tools for this are on the one hand, from a local point of view, a notion of "first fundamental form" independent of an ambient space $\mathbb{R}^{n+1}$ (similar to the notion of intrinsic geometry in the previous chapter), and on the other hand, from a global point of view, the notion of a "manifold". The local notion goes back essentially to the famous lecture of Riemann[1], which explains the modern notions *Riemannian geometry*, *Riemannian manifold* and *Riemannian space*. From the point of view of the development in the book up to now, this is motivated on the one hand by the intrinsic geometry of surfaces, including the Gauss-Bonnet theorem, and on the other hand by the natural occurrence of such spaces which can *not* in any meaningful way be embedded as hypersurfaces in some $\mathbb{R}^n$, as for example the Poincaré upper half-plane as a model of non-Euclidean geometry. Furthermore, the space-times of $3+1$ dimensions which are considered in general relativity do not admit an ambient space in a natural way. This motivates the intention of explaining all geometric quantities in a purely intrinsic manner.

[1]B. Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, edited by H. Weyl, Springer, 1921; see also [**7**], Vol. II, Chapter 4.

In the previous Chapters 3 and 4 we have basically been considering surface elements $f\colon U \to \mathbb{R}^{n+1}$, where $U \subset \mathbb{R}^n$ was a given open set. From a geometric point of view, we are really more interested in the image set $f(U)$ than we are in the map f itself. Nonetheless, for a description and for local calculations we do use the parameter set U and the parametrization f:

$$U \ni u \overset{f}{\longmapsto} p = f(u) \in f(U).$$

If we decide that the basic object we are considering is the image $f(U)$, then we come to view the inverse mapping

$$f(U) \ni p \overset{f^{-1}}{\longmapsto} u \in U$$

as an image which is "thrown" from $f(U)$, in order to carry out calculations in U. This map is called a "chart" in what follows, which should be thought of as creating a "map" (but the word "map" has a fixed, different meaning in mathematics, so that one uses "chart" instead), and a set of charts which cover the object of interest forms an "atlas", just as a world atlas contains a map containing an arbitrary location on the earth. For the mathematical notion this means that every point has a neighborhood which is contained in one of the charts, in which local computations near that point can be carried out in the corresponding set U. What we have to be able to guarantee is that all defined notions are *independent* of the choice of charts used, just as the Gaussian curvature in the theory of surfaces was independent of the parametrization. In particular, we need to carefully consider the transformations which map us from one chart into a different, nearby one.

5A The notion of a manifold

We have already met submanifolds of $\mathbb{R}^n$ in the form of zero sets of differentiable maps, cf. Chapter 1. If there is no ambient space to begin with, this definition no longer makes any sense. Instead, one uses a description in terms of local coordinates in the form of parametrizations or *charts*, just as one considers maps of the earth to approximate that round object by flat pictures. Note that the chart maps go in the opposite direction from the usual parametrization we have been using up to now.

5.1. Definition. (Abstract differentiable manifold)
A *k-dimensional differentiable manifold* (briefly: a *k-manifold*) is a set M together with a family $(M_i)_{i\in I}$ of subsets such that

1. $M = \bigcup_{i\in I} M_i$ (union),
2. for every $i \in I$ there is an injective map $\varphi_i \colon M_i \to \mathbb{R}^k$ so that $\varphi_i(M_i)$ is open in $\mathbb{R}^k$, and
3. for $M_i \cap M_j \neq \emptyset$, $\varphi_i(M_i \cap M_j)$ is open in $\mathbb{R}^k$, and the composition
$$\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(M_i \cap M_j) \to \varphi_j(M_i \cap M_j)$$
is differentiable for arbitrary i, j.

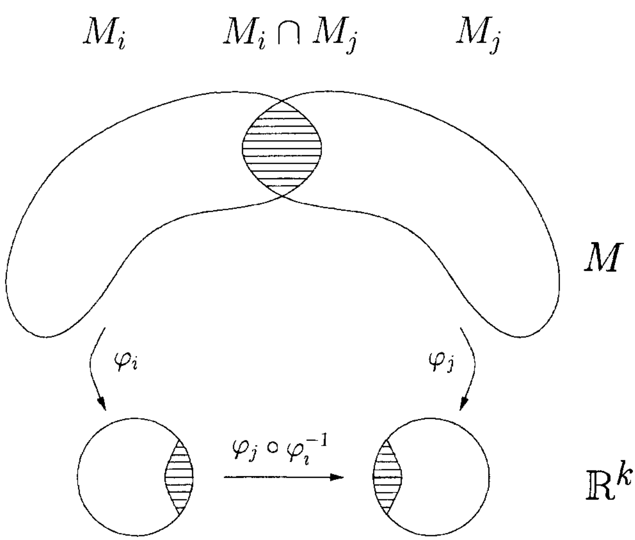

Figure 5.1. Charts on a manifold

Each φ_i is called a *chart*, φ_i^{-1} is referred to as the *parametrization*, the set $\varphi_i(M_i)$ is called the *parameter domain*, and $(M_i, \varphi_i)_{i\in I}$ is called an *atlas*. The maps $\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(M_i \cap M_j) \to \varphi_j(M_i \cap M_j)$, defined on the intersections of two such charts, are called *coordinate transformations* or *transition functions.* Without restriction of generality, we may assume that the atlas is *maximal* with respect to adding more charts satisfying the conditions 2 and 3 above. A maximal atlas in this sense is then referred to as a *differentiable structure.*

EXAMPLES:

1. Every open subset U of $\mathbb{R}^k$ is a k-manifold, where a single chart is sufficient for the entire manifold, namely the inclusion map $\varphi\colon U \to \mathbb{R}^k$. Condition 3 is trivially satisfied in this case.
2. Every k-dimensional submanifold M of $\mathbb{R}^n$ (cf. Chapter 1) is also a k-dimensional manifold in the sense of the above definition. If M is given locally by $M = \{x \in \mathbb{R}^n \mid F(x) = 0\}$, where $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a continuously differentiable submersion (i.e., the differential DF is surjective, or in other words $\mathrm{Rank}(DF) = n - k$), then according to the implicit functions theorem one can locally solve the equation
$$F(x^1, \ldots, x^n) = 0$$
(perhaps after a renumbering) in the explicit form
$$\begin{aligned} x^{k+1} &= x^{k+1}(x^1, \ldots, x^k), \\ &\vdots \\ x^n &= x^n(x^1, \ldots, x^k). \end{aligned}$$
By making the association
$$(x^1, \ldots, x^k) \longmapsto (x^1, \ldots, x^k, x^{k+1}, \ldots, x^n),$$
we get a parametrization, while the association $(x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^k)$ gives us a chart.
3. The (abstract) torus $\mathbb{R}^2/\mathbb{Z}^2$ is defined as the quotient (group) of these two Abelian groups. To give it a differentiable structure, one defines charts by starting with arbitrary open sets M_i in $\mathbb{R}^2$ (more precisely, take their images in the quotient) which are contained in the open square $(x_0 - \frac{1}{2}, x_0 + \frac{1}{2}) \times (y_0 - \frac{1}{2}, y_0 + \frac{1}{2})$ for an arbitrary point $(x_0, y_0) \in \mathbb{R}^2$. Then set $\varphi(x, y) := (x - x_0, y - y_0)$ to obtain one chart (depending on the choice of (x_0, y_0)). It

follows that the coordinate transformations are just translations in $\mathbb{R}^2$. One sees without difficulty that three of these charts suffice to cover the image, namely the just mentioned squares centered at the points $(0,0)$, $(\frac{1}{3}, \frac{1}{3})$, $(\frac{2}{3}, \frac{2}{3})$. Two such sets do *not* suffice.

Similar results, with appropriate modifications, hold also for the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$.

4. The (abstract) *Klein bottle* is a quotient of the two-dimensional torus by the involution $(x,y) \mapsto (x + \frac{1}{2}, -y)$. We may take any square in the (x,y)-plane whose length in the x-direction is at most $\frac{1}{2}$ and whose length in the y-direction is at most 1, as charts.

5. The real projective plane is the quotient of the two-sphere

$$\mathbb{R}P^2 := S^2/\sim,$$

where the equivalence relation is given by $x \sim -x$. We may take any open set in S^2 as M_i, provided it is contained in a hemisphere (by which we mean half a sphere), and in particular contains no antipodal points. φ can be defined as a projection to a hemisphere, followed by a projection of this onto a disc.
A model of this is the closed disc modulo the identification of the antipodal pairs of points on the boundary. On the other hand, the "classical" model of projective geometry is all of $\mathbb{R}^2$ with an added "line at infinity".
An atlas of the projective plane containing three charts can be constructed as the charts induced by the centrally symmetric atlas on S^2, which consists of the six hemispheres in the three directions (x^1, x^2, x^3).

6. The *rotation group* $\mathbf{SO}(3)$ is defined as the set of all (real) orthogonal (3×3)-matrices with determinant equal to 1. We show that it is a 3-manifold by defining the *Cayley map*

$$CAY\colon \mathbb{R}^3 \to \mathbf{SO}(3), \quad CAY(A) := (\mathbf{1} + A)(\mathbf{1} - A)^{-1}.$$

Here $\mathbf{1}$ denotes the unit matrix, and A denotes the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

with real parameters a, b, c, which can also be viewed as an element of $\mathbb{R}^3$. The Cayley map is injective, and the inverse map can be used as a chart of $\mathbf{SO}(3)$ and determined as follows:

$$CAY(A) = B \Longleftrightarrow B(\mathbf{1} - A) = \mathbf{1} + A$$

$$\Longleftrightarrow (B + \mathbf{1})A = B - \mathbf{1} \Longleftrightarrow A = (B + \mathbf{1})^{-1}(B - \mathbf{1}).$$

Note that $B + \mathbf{1}$ is always invertible, except when -1 is an eigenvalue of B. The matrices B for which this last condition holds are precisely the rotations by π. In fact, the image of the Cayley map is all of $\mathbf{SO}(3)$ with the exception of the set of rotation matrices by a rotational angle of π.
The set of all such rotations by π is naturally bijective to the set of all possible axes of rotation, hence bijective to a projective plane $\mathbb{R}P^2$. To get charts covering this exceptional set of $\mathbf{SO}(3)$, we require three more charts, just as in the above example of an atlas for the projective plane. If we define $\mathbf{E_i}$ as the rotation matrix by an angle of π around the ith axis, and if we formally set $\mathbf{E_0} = \mathbf{1}$, then the following four maps (resp. their inverses) define an atlas of $\mathbf{SO}(3)$:[2]

$$A \mapsto \mathbf{E_i} \cdot CAY(A), \quad i = 0, 1, 2, 3.$$

The four parametrizations of the atlas thus consist of the Cayley maps "centered at" $\mathbf{1}, \mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$. The transformations from one chart to another are given by matrix multiplication and are therefore differentiable.

[2] I am indebted to Prof. E. Grafarend for a question giving rise to this, which arose from applications in geodesy. Traditionally one considers in geodesy only a single chart for the rotation group, yielding the *Euler angles* or *Cardan angles*.

5.2. Definition. (Structures on a manifold)

Given a k-dimensional manifold, one gets *additional structure* by placing additional requirements on the transformation functions $\varphi_j \circ \varphi_i^{-1}$, which belong to the atlas of the manifold; if all $\varphi_j \circ \varphi_i^{-1}$ are (left-hand side), then one speaks of (right-hand side) as follows:

continuous	$\leftrightarrow$	topological manifold
differentiable	$\leftrightarrow$	differentiable manifold
C^1-differentiable	$\leftrightarrow$	C^1-manifold
C^r-differentiable	$\leftrightarrow$	C^r-manifold
C^∞-differentiable	$\leftrightarrow$	C^∞-manifold
real analytic	$\leftrightarrow$	real analytic manifold
complex analytic	$\leftrightarrow$	complex manifold of dimension $\frac{k}{2}$
affine	$\leftrightarrow$	affine manifold
projective	$\leftrightarrow$	projective manifold
conformal	$\leftrightarrow$	manifold with a conformal structure
orientation-preserving	$\leftrightarrow$	orientable manifold

Convention: In what follows we shall understand by the term "manifold" a C^∞-manifold, and "differentiable" will always mean C^∞. One can show that a C^k-atlas always contains a C^∞ one, so that this convention is not a real restriction.

5.3. Definition. (Topology)

A subset $O \subseteq M$ is called *open*, if $\varphi_i(O \cap M_i)$ is open in $\mathbb{R}^k$ for every i. This defines a *topology* on M as the set of all open sets. Then all φ_i are continuous, since the inverse images under them of open sets are again open. M is said to be *compact*, if every open covering contains a finite sub-covering (Heine-Borel covering property). In particular, every compact manifold can be covered with finitely many charts.

Running assumption: In what follows we will always assume that the manifolds which occur satisfy the *Hausdorff separation axiom* (T_2-axiom), formulated as follows. Every two distinct points p, q have disjoint open neighborhoods U_p, U_q. Note that this property does not follow from Definition 5.1.

The important point here is that *locally* (or *in the small*) the topology of a manifold is the same as that of an $I\!R^k$. In particular this means that the inverse images of open ε-balls in $I\!R^k$ are again open in M, although one cannot necessarily make sense of the notion of ε-balls there, as there is no distance function (metric) defined. But this suffices to define the notion of convergence of sequences just as in $I\!R^k$. In addition, the topology of every manifold is locally compact, which means that every point has a compact neighborhood, for example the inverse image of a closed ε-ball in $I\!R^k$.

5.4. Definition. (Differentiable map)
Let M be an m-dimensional differentiable manifold, and let N be an n-dimensional differentiable manifold; furthermore, let $F\colon M \to N$ be a given map. F is said to be *differentiable*, if for all charts $\varphi : U \to I\!R^m, \psi : V \to I\!R^n$ with $F(U) \subset V$ the composition $\psi \circ F \circ \varphi^{-1}$ is also differentiable.

In particular this defines the concept of a *differentiable function* $f\colon M \to I\!R$, where in this case $I\!R$ carries the (identity) standard chart.

This definition is independent of the choice of φ and ψ. A *diffeomorphism* $F : M \to N$ is defined to be a bijective map which is differentiable in both directions. One then calls the two manifolds M and N *diffeomorphic*. Two diffeomorphic manifolds necessarily have the same dimension. This is because for $I\!R^m$ and $I\!R^n$ with $n \neq m$, there is no bijective mapping which is differentiable in both directions, since the corresponding Jacobi matrix is not square and hence cannot have non-vanishing determinant (i.e., cannot be invertible).

REMARK: With respect to additional structures on our manifold, one can similarly define when a map is analytic or complex analytic or affine, etc. For example, let us consider here the *Riemann sphere* $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. By means of the inclusion $\mathbb{C} \to \widehat{\mathbb{C}}$ one has a chart, and a second is given by $z \mapsto \frac{1}{z}$. These two charts define a *complex structure* on the Riemann sphere, if one adds all compatible charts. Then all meromorphic maps of the Riemann sphere to itself are differentiable maps in the sense of the above definition, for example, also the map $z \mapsto z^{-k}$. Furthermore, this defines a conformal structure on S^2 since every complex-analytic function $f(z)$ with $f' \neq 0$ in one variable z is conformal, cf. Section 3D.

Convention: For a chart φ we will denote by $(u^1, \ldots, u^k)$ the standard coordinates of $\mathbb{R}^k$, and by $(x^1, \ldots, x^k)$ the corresponding coordinates in M. Thus, $x^i(p)$ is the function given by the ith coordinate of $\varphi(p)$, $x^i(p) = u^i(\varphi(p))$. The functions $(u^1, \ldots, u^k)$ as well as $(x^1, \ldots, x^k)$ are thus on the one hand the coordinates of the points considered, while on the other hand $(u^1, \ldots, u^k)$ and $(x^1, \ldots, x^k)$ are also viewed as variables, with respect to which we can form derivatives. For a real-valued function $f : M \to \mathbb{R}$ we set

$$\left.\frac{\partial f}{\partial x^i}\right|_p := \left.\frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\right|_{\varphi(p)}$$

and emphasize this notation by thinking of the partial derivatives as infinitesimal changes of a function in the directions x^i or u^i.

5B The tangent space

Let M be an n-dimensional differentiable manifold and $p \in M$ a fixed point. The tangent space of M at the point p is going to be thought of as the n-dimensional set of "directional vectors", which – starting at p – point in all directions of M, cf. for example [**27**]. Since there is no ambient space, this notion has to be intrinsically defined and constructed. For this, there are three possible definitions, all of which we describe here.

5.5. Definition. (Tangent vector, tangent space)
Geometric Definition:
A *tangent vector* at p is an equivalence class of differentiable curves $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$, where $c \sim c^* \Leftrightarrow (\varphi \circ c)\dot{}(0) = (\varphi \circ c^*)\dot{}(0)$ for every chart φ containing p.

Briefly: *tangent vectors are tangents to curves lying on the manifold.*

Unfortunately there is no privileged representative of such an equivalence class, and such a representative would depend on the choice of chart (for example, a line in the parameter domain).

Algebraic Definition:
A *tangent vector* X at p is a derivation (derivative operator) defined on the set of *germs of functions*

$$\mathcal{F}_p(M) := \{f : M \to I\!R \mid f \text{ differentiable}\}/\sim \quad ,$$

where the equivalence relation $\sim$ is defined by declaring $f \sim f^*$ if and only if f and f^* coincide in a neighborhood of p. The value $X(f)$ is also referred to as the *directional derivative* of f in the direction X.

This definition means more precisely the following. X is a map $X : \mathcal{F}_p(M) \to I\!R$ with the two following properties:

1. $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$, $\alpha, \beta \in I\!R$, $f, g \in \mathcal{F}_p(M)$ (*$I\!R$-linearity*);
2. $X(f \cdot g) = X(f) \cdot g(p) + f(p) \cdot X(g)$ for $f, g \in \mathcal{F}_p(M)$ (*product rule*).
   (For this to make sense, both f and g have to be defined in a neighborhood of p.)

Briefly: *tangent vectors are derivations acting on scalar functions.*

Physical Definition:
A *tangent vector* at the point p is defined as an n-tuple of real numbers $(\xi^i)_{i=1,\dots,n}$ in a coordinate system $x^1, \dots, x^n$ (that is, in a chart), in such a way that in any other coordinate system $\widetilde{x}^1, \dots, \widetilde{x}^n$ (i.e., in any other chart) the same vector is given by a corresponding n-tuple $(\widetilde{\xi}^i)_{i=1,\dots,n}$, where

$$\widetilde{\xi}^i = \sum_j \frac{\partial \widetilde{x}^i}{\partial x^j}\bigg|_p \xi^j.$$

Briefly: *tangent vectors are elements of $I\!R^n$ with a particular transformation behavior.*

The *tangent space* T_pM of M at p is defined in all cases as the set of all tangent vectors at the point p. By definition T_pM and T_qM are disjoint if $p \neq q$.

For the special case of an open subset $U \subset I\!R^n$, the tangent space can be identified with $T_pU := \{p\} \times I\!R^n$ endowed with the standard basis $(p, e_1), \dots, (p, e_n)$. The vector e_i corresponds to the curve $c_i(t) := p + t \cdot e_i$ (geometric definition) and to the derivation given by the partial derivative $f \longmapsto \frac{\partial f}{\partial u^i}\big|_p$ (algebraic definition). Therefore 5.5 is compatible with the previous definitions given in 1.7 and 3.1. The directional derivative coincides in $I\!R^n$ with the directional derivative which was already defined in 4.1.

Special (geometric) tangent vectors are those given by the parameter lines (lines along which parameter values are constant), formally meaning the equivalence classes of them. The corresponding special tangent vectors in the algebraic definition are the partial derivatives $\frac{\partial}{\partial x^i}\big|_p$ defined by

$$\frac{\partial}{\partial x^i}\bigg|_p (f) := \frac{\partial f}{\partial x^i}\bigg|_p = \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\bigg|_{\varphi(p)}$$

in a chart φ which contains p. As a notational convenience one also writes $\partial_i\big|_p$ instead of $\frac{\partial}{\partial x^i}\big|_p$. The special tangent vectors in the sense of the physical definition are in this case simply the tuples which consist of zeros except in the ith place.

The geometric definition is probably the most intuitive (*a tangent vector is a tangent to a curve*), but not easy to work with. In this definition it is not even clear that the tangent space is a real vector space. The algebraic definition is most convenient for doing computations, and by its very definition it is independent of any chart. The physical definition will be further clarified below. The art of doing computations with the geometric quantities of the physical definition goes back to G. Ricci and is called the *Ricci calculus*, cf. [**16**]. A vector is simply written as ξ^i, and the very fact that the notion involves a superscript indicates the transformation behavior, in this case, for example, as a vector (or 1-contravariant tensor), cf. Section 6.1. This aspect will be of importance in what follows, but for all definitions we will give a coordinate-independent formulation as far as this is feasible. The equivalence of these three definitions is explained for example in [**39**], Chapter 2. In what follows we base our analysis on the algebraic definition and will therefore not require this equivalence.

5.6. Theorem. The (algebraic) tangent space at p on an n-dimensional differentiable manifold is an n-dimensional $\mathbb{R}$-vector space and is spanned in any coordinate system $x^1, \ldots, x^n$ in a given chart by

$$\frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p.$$

For every tangent vector X at p one has

$$X = \sum_{i=1}^{n} X(x^i) \frac{\partial}{\partial x^i}\bigg|_p.$$

Looking at the last equation, we see that the components ξ^i of a tangent vector X in the Ricci calculus are nothing but the $X(x^i)$, that is, the directional derivatives of the coordinate functions x^i in the direction X. To prove the statement of the theorem we require the following lemma.

5.7. Lemma. If X is a tangent vector and f is a constant function, then $X(f) = 0$.

PROOF: First suppose $f = 1$ everywhere. Then by the product rule 5.5.2 we have

$$X(1) = X(1 \cdot 1) = X(1) \cdot 1 + 1 \cdot X(1) = 2 \cdot X(1),$$

hence $X(1) = 0$. Now suppose that f has the constant value $f = c$. Then by the linearity 5.5.1 we have

$$X(c) = X(c \cdot 1) = c \cdot X(1) = c \cdot 0 = 0.$$

$\square$

PROOF OF 5.6: The proof utilizes an adapted representation of the transition functions in local coordinates. We calculate in a chart $\varphi : U \longrightarrow V$, where without restricting generality we may assume V is an open ε-ball with $\varphi(p) = 0$, hence $x^1(p) = \cdots = x^n(p) = 0$. Let $h : V \to I\!R$ be a differentiable function, and $f := h \circ \varphi$. We set

$$h_i(y) := \int_0^1 \frac{\partial h}{\partial u^i}(t \cdot y)dt \quad (\text{note: } h \in C^\infty \Rightarrow h_i \in C^\infty)$$

and perform the following computation:

$$\sum_{i=1}^n \frac{\partial h}{\partial u^i}(t \cdot y) \cdot \underbrace{\frac{d(tu^i)}{dt}}_{=u^i} = \frac{\partial h}{\partial t}(t \cdot y),$$

which implies

$$\sum_{i=1}^n h_i(y) \cdot u^i = \int_0^1 \frac{\partial h}{\partial t}(t \cdot y)dt = h(y) - h(0).$$

From this we get, using the identities $f = h \circ \varphi$, $f_i = h_i \circ \varphi$, $x^i = u^i \circ \varphi$, the equation

$$f(q) - f(p) = \sum_{i=1}^n f_i(q) \cdot x^i(q)$$

for a variable point q. Taking derivatives, we get

$$\left.\frac{\partial f}{\partial x^i}\right|_p = f_i(p).$$

Now if we are given a tangent vector X at p, then it follows from properties 1 and 2 in 5.5 that

$$X(f) = X\Big(f(p) + \sum_{i=1}^{n} f_i x^i\Big) = 0 + \sum_{i=1}^{n} X(f_i)\cdot \underbrace{x^i(p)}_{=0} + \sum_{i=1}^{n} f_i(p)\cdot X(x^i)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_p \cdot X(x^i) = \Big(\sum_{i=1}^{n} X(x^i)\cdot \frac{\partial}{\partial x^i}\Big|_p\Big)(f)$$

for every f. It remains to show that the vectors $\frac{\partial}{\partial x^i}\big|_p$ are linearly independent. But this is easy to see, since $\frac{\partial}{\partial x^i}\big|_p(x^j) = \frac{\partial x^j}{\partial x^i} = \delta_i^j$.

Note that this proof only works for C^∞-manifolds, as otherwise the degree of differentiability of h_i is one less that of h. In fact, the algebraic tangent space on a C^k-manifold is infinite-dimensional. But there are no difficulties in simply passing to the subspace spanned by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ and performing the same calculations there. □

5.8. Definition and Lemma. (Derivative, chain rule)
Let $F : M \to N$ be a differentiable map, and let p, q be two fixed points with $F(p) = q$. Then the *derivative* or the *differential* of F at p is defined as the map

$$DF|_p : T_pM \longrightarrow T_qN$$

whose value at $X \in T_pM$ is given by $(DF|_p(X))(f) := X(f \circ F)$ for every $f \in \mathcal{F}_q(N)$ (which automatically implies the relation $f \circ F \in \mathcal{F}_p(M)$). For the derivative as defined in this manner, one has the *chain rule* in the form

$$D(G \circ F)|_p = DG|_{F(p)} \circ DF|_p$$

for every composition $M \xrightarrow{F} N \xrightarrow{G} Q$ of maps, or, more briefly, $D(G \circ F) = DG \circ DF$.

PROOF: By definition we have

$$D(G \circ F)|_p(X)(f) = X(f \circ G \circ F)$$

$$= (DF|_p(X))(f \circ G) = \Big(DG|_q\big(DF|_p(X)\big)\Big)(f).$$

□

REMARK: One can view $DF|_p$ as a linear approximation of F at p, just as in vector analysis on $\mathbb{R}^n$. In coordinates $x^1, \ldots, x^m$ on M and $y^1, \ldots, y^n$ on N, $DF|_p$ is represented by the Jacobi matrix, for which we have the more precise relation

$$DF|_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \sum_i \frac{\partial(y^i \circ F)}{\partial x^j}\Big|_p \frac{\partial}{\partial y^i}\Big|_q.$$

In the physical definition of tangent spaces, the chain rule consists essentially of the product of the Jacobi matrices, applied to the tangent vector. In the geometric definition of the tangent space (i.e., for equivalence classes of curves through the point p), the differential is simply described by the transport of curves, as follows:

$$DF|_p([\, c\,]) := [\, F \circ c\,],$$

and the chain rule $DG\big(DF([c])\big) = [G \circ F \circ c]$ is then quite obvious. Note the action on the tangent of a curve:

$$\dot{c}(0) \mapsto (F \circ c)^{\cdot}(0) = DF|_p(\dot{c}(0)).$$

EXAMPLES:

(i) In case $F : U \to \mathbb{R}^{n+1}$ ($U \subset \mathbb{R}^n$) is a surface element in the sense of Chapter 3 with $u \mapsto F(u) = p$, then the differential of F acts in the following way on the basis $\frac{\partial}{\partial u^1}\big|_u, \ldots, \frac{\partial}{\partial u^n}\big|_u$ of T_uU resp. $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^{n+1}}\big|_p$ of $T_p\mathbb{R}^{n+1}$:

$$DF\Big|_u\left(\frac{\partial}{\partial u^j}\Big|_u\right) = \sum_i \frac{\partial x^i}{\partial u^j}\Big|_u \cdot \frac{\partial}{\partial x^i}\Big|_p,$$

where the matrix $\frac{\partial x^i}{\partial u^j}$ is the familiar *Jacobi matrix* of the mapping F. Here, x^i is the ith component of $F(u^1, \ldots, u^n)$, also written as the function $x^i(u^1, \ldots, u^n)$.

(ii) If $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ are two coordinate systems on a single manifold, then one has similarly, for F equal to the identity,

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}.$$

(iii) For the components ξ^i and η^j, respectively, of a tangent vector $X = \sum_j \xi^j \frac{\partial}{\partial x^j} = \sum_i \eta^i \frac{\partial}{\partial y^i}$, one has similarly $X = \sum_j \xi^j \frac{\partial}{\partial x^j} = \sum_{i,j} \xi^j \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$; hence $\eta^i = \sum_j \xi^j \frac{\partial y^i}{\partial x^j}$. This is precisely the transformation behavior of tangent vectors in Ricci calculus (Definition 5.5).

The following *summation convention* is used in Ricci calculus, and is usually referred to as the Einstein summation convention: sums are formed over indices which occur in formulas as both an upper (in the numerator) and a lower (in the denominator) subscript, without the explicit summation symbol, for example

$$h_{ik} = h_i^j g_{jk} \quad \text{instead of} \quad h_{ik} = \sum_j h_i^j g_{jk} \text{ and}$$

$$\eta^i = \xi^j \frac{\partial y^i}{\partial x^j} \quad \text{instead of} \quad \eta^i = \sum_j \xi^j \frac{\partial y^i}{\partial x^j}.$$

5.9. Definition. (Vector field)
A differentiable *vector field* X on a differentiable manifold is an association $M \ni p \longmapsto X_p \in T_pM$ such that in every chart $\varphi\colon U \to V$ with coordinates $x^1, \ldots, x^n$, the coefficients $\xi^i\colon U \to I\!R$ in the representation (valid at a point)

$$X_p = \sum_{i=1}^n \xi^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

are differentiable functions.

Another common notation for this is $X = \sum_i \xi^i \frac{\partial}{\partial x^i}$ or, in Ricci calculus, $X = \xi^i$. Note that in the physical definition, a vector field is identified with the n-tuple $(\xi^1, \ldots, \xi^n)$ of functions of the coordinates $x^1, \ldots, x^n$.

As to the notations used in conjunction with vector fields, for a scalar function $f : M \to I\!R$, the symbol fX denotes the vector field $(fX)_p := f(p) \cdot X_p$ (one can say that *the set of vector fields is a module over the ring of functions f on M*), while the symbol $Xf = X(f)$

denotes the function $(Xf)(p) := X_p(f)$ (in other words, Xf is the derivative of f in the direction of X).

5C Riemannian metrics

The first fundamental form of a surface element is a scalar product, which is defined by restricting the Euclidean scalar product to each tangent space, as we have explained in Chapter 3. In our present endeavor, we have to find a way to do this without the ambient space, that is, defining (intrinsically) a scalar product on each tangent space. Recall the following fact from linear algebra, which we will require in this regard.

The space $L^2(T_pM; I\!R) = \{\alpha\colon T_pM \times T_pM \to I\!R \mid \alpha \text{ bilinear}\}$ has the basis

$$\{dx^i|_p \otimes dx^j|_p \mid i,j = 1,\ldots,n\},$$

where the dx^i form the *dual basis* in the dual space

$$(T_pM)^* = L(T_pM; I\!R),$$

defined as follows:

$$dx^i|_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The bilinear forms $dx^i|_p \otimes dx^j|_p$ are defined in terms of their action on the basis (this action being then extended by linearity):

$$(dx^i|_p \otimes dx^j|_p)\Big(\frac{\partial}{\partial x^k}\Big|_p, \frac{\partial}{\partial x^l}\Big|_p\Big) := \delta^i_k\delta^j_l = \begin{cases} 1 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

By inserting the basis, for the coefficients of the representation

$$\alpha = \sum_{i,j} \alpha_{ij} \cdot dx^i \otimes dx^j$$

one obtains the expression

$$\alpha_{ij} = \alpha\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big).$$

In Ricci calculus, the form α is just represented by the symbol α_{ij}; one also refers to this as a *tensor of degree two*, cf. 6.1.

5.10. Definition. (Riemannian metric)
A *Riemannian metric* g on M is an association $p \longmapsto g_p \in L^2(T_pM; \mathbb{R})$ such that the following conditions are satisfied:

1. $g_p(X,Y) = g_p(Y,X)$ for all X, Y, (*symmetry*)
2. $g_p(X,X) > 0$ for all $X \neq 0$, (*positive definiteness*)
3. The coefficients g_{ij} in every local representation (i.e., in every chart)
$$g_p = \sum_{i,j} g_{ij}(p) \cdot dx^i|_p \otimes dx^j|_p$$
are differentiable functions. (*differentiability*)

The pair (M, g) is then called a *Riemannian manifold.* One also refers to the Riemannian metric as the *metric tensor.* In local coordinates the metric tensor is given by the matrix (g_{ij}) of functions. In Ricci calculus this is simply written as g_{ij}.

REMARKS:

1. A Riemannian metric g defines at every point p an *inner product* g_p on the tangent space T_pM, and therefore the notation $\langle X, Y\rangle$ instead of $g_p(X,Y)$ is also used. The notions of angles and lengths are determined by this inner product, just as these notions are determined by the first fundamental form on surface elements. The length or norm of a vector X is given by $||X|| := \sqrt{g(X,X)}$, and the angle β between two tangent vectors X and Y can be defined by the validity of the equation $\cos\beta \cdot ||X|| \cdot ||Y|| = g(X,Y)$, cf. Chapter 1.

2. If the condition that g is positive definite is replaced by the weaker condition that it is *non-degenerate* (meaning that $g(X,Y) = 0$ for all Y implies $X = 0$), then one arrives at the notion of a *pseudo-Riemannian metric* or *semi-Riemannian metric*, in which all notions are defined in exactly the same way as for a Riemannian metric. In particular, a so-called *Lorentzian metric* is defined as one for which the signature of g is $(-,+,+,+)$; such metrics are basic to the general theory of relativity. In this case the tangent spaces are modeled after

Minkowski space $\mathbb{R}^4_1$ instead of Euclidean space (cf. Section 3E) with the metric

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The difference compared with Euclidean space is that there are vectors $X \neq 0$ with $g(X, X) = 0$, so-called *null vectors.* We have already studied the three-dimensional Minkowski space in connection with curves and surfaces (compare sections 2E and 3E). The tensor g_{ij} is referred to in the theory of relativity as the *gravitational potential* or *gravitational field*, cf. [**25**], Section 1.3. It gives a metric form to the manifold (four-dimensional space-time) according to the gravity coming from the matter which is contained in the space.

Examples:

(i) The first fundamental form g of a hypersurface element in $\mathbb{R}^{n+1}$ is an example of a Riemannian metric.

(ii) The standard example is $(M, g) = (\mathbb{R}^n, g_0)$, where the metric $(g_0)_{ij} = \delta_{ij}$ (identity matrix) is the Euclidean metric in the standard chart of $\mathbb{R}^n$ (given by Cartesian coordinates). This space is also referred to as *Euclidean space* and denoted by $\mathbb{E}^n$. The metric is

$$(g_0)_{ij} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

so that $g_0(.,.) = \langle \cdot, \cdot \rangle$ is, not unexpectedly, nothing but the *Euclidean inner product.*

(iii) A different Riemannian metric on $\mathbb{R}^n$ is given for example by $g_{ij}(x_1, \dots, x_n) := \delta_{ij}(1 + x_i x_j)$:

$$(g_{ij}) = \begin{pmatrix} 1 + x_1^2 & 0 & \dots & 0 \\ 0 & 1 + x_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 + x_n^2 \end{pmatrix}$$

Similarly, one can define numerous Riemannian metrics simply by choosing the coefficients g_{ij} arbitrarily, provided only that one has positive definiteness or non-degeneracy of the metric.

(iv) After choosing constants $0 < b < a$, on $(0, 2\pi) \times (0, 2\pi) \subset \mathbb{R}^2$, $0 < r < 1$, one can define a Riemannian metric by

$$(g_{ij}(u,v)) = \begin{pmatrix} b^2 & 0 \\ 0 & (a + b\ \cos u)^2 \end{pmatrix}.$$

This coincides with the first fundamental form on an open subset of the torus of revolution (cf. Chapter 3).

(v) We can give the abstract torus $\mathbb{R}^2/\mathbb{Z}^2$ a uniquely defined Riemannian metric g with the property that the projection

$$(\mathbb{R}^2, g_0) \longrightarrow (\mathbb{R}^2/\mathbb{Z}^2, g)$$

is a local isometry in the sense of 5.11. This is called the *flat torus*. In the chart $(0,1) \times (0,1)$ the metric is $(g_{ij}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, as in the Euclidean plane.

(vi) Similarly, the real projective plane $\mathbb{R}P^2 = S^2/\pm$ can be given a unique Riemannian metric g such that the projection $(S^2, g_1) \to (\mathbb{R}P^2, g)$ is a local isometry in the sense of 5.11, where g_1 is the standard metric on the unit sphere.

(vii) The Poincaré upper half-plane $\{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ with the metric

$$(g_{ij}(x,y)) := \frac{1}{y^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a Riemannian manifold. In this metric, length is given by $||\frac{\partial}{\partial y}|| = \frac{1}{y}$; thus the half-lines in the y-direction have infinite length: $\int_\eta^1 \frac{1}{t}dt = -\log(\eta) \longrightarrow \infty$ for $\eta \to 0$ and $\int_1^\eta \frac{1}{t}dt = \log(\eta) \longrightarrow \infty$ for $\eta \to \infty$. In fact, every geodesic is of infinite length in both directions. We refer also to the Exercises at the end of Chapter 4 as well as Section 7A for more details.

5.11. Definition. (Maps which are compatible with the metric) A differentiable map $F\colon M \longrightarrow \widetilde{M}$ between two Riemannian manifolds $(M,g), (\widetilde{M},\widetilde{g})$ is called a *(local) isometry*, if for all points p and tangent vectors X, Y we have

$$\widetilde{g}_{F(p)}(DF|_p(X), DF|_p(Y)) = g_p(X,Y);$$

more generally, F is called a *conformal mapping*, if there is a function $\lambda\colon M \to \mathbb{R}$ without zeros, such that for all p, X, Y, one has

$$\widetilde{g}_{F(p)}(DF_p(X), DF_p(Y)) = \lambda^2(p) g_p(X,Y).$$

See also Definitions 3.29 and 4.29.

By definition a local isometry preserves the length of a vector, angles, and areas and volumes, whereas a conformal mapping preserves angles but rescales the length of any vector by the factor λ.

EXAMPLES: The map $(x,y) \mapsto (\cos x, \sin x, y)$ is a local isometry of the plane onto a cylinder. Stereographic projection defines a conformal map between the plane and the punctured sphere.

QUESTION: Does there exist a Riemannian metric on an arbitrary manifold M? Locally there is no problem in constructing one, as we choose any (g_{ij}) which is both positive definite and symmetric. To make this construction global, one can use the method of a *partition of unity*. To introduce this notion, we define the following

NOTATION: For a given function $f\colon M \to \mathbb{R}$, the topological closure

$$supp(f) := \overline{\{x \in M \mid f(x) \neq 0\}}$$

is called the *support of f*.

5.12. Definition and Lemma. (Partition of unity)

A differentiable *partition of unity* on a differentiable manifold M is a family $(f_i)_{i\in I}$ of differentiable functions $f_i\colon M \to \mathbb{R}$ such that the following conditions are satisfied:

1. $0 \leq f_i \leq 1$ for all $i \in I$,
2. every point $p \in M$ has a neighborhood which intersects only finitely many of the $supp(f_i)$, and

3. $\sum_{i \in I} f_i \equiv 1$ (locally this is always to be a finite sum).

If there is a partition of unity on M such that the support $supp(f_i)$ of each function is contained in a coordinate neighborhood, then there exists a Riemannian metric on M.

PROOF: For each $i \in I$ choose $g_{kl}^{(i)}$ as a symmetric, positive definite matrix-valued function (in the chart associated with $supp(f_i)$). This locally defines a Riemannian metric $g^{(i)}$, and $f_i \cdot g^{(i)}$ is differentiable and well-defined on all of M, namely, it vanishes identically outside of $supp(f_i)$. Then we set

$$g := \sum_{i \in I} f_i \cdot g^{(i)}.$$

It follows that g is symmetric and positive semi-definite because $f_i \geq 0$ and $g^{(i)} > 0$, and from $\sum_i f_i \equiv 1$ we see that g is even positive definite at every point. □

WARNING: The same method does *not* show the existence of an indefinite metric $\widetilde{g}$ on M, because in this case $\tilde{g}$ can degenerate, even if all $\widetilde{g}^{(i)}$ are non-degenerate. In fact, there are topological obstructions to the existence of indefinite metrics. For example there is a Lorentz metric of type $(- + + \cdots +)$ on a compact manifold if and only if the Euler characteristic satisfies $\chi = 0$. This is because precisely in this case, a line element field exists[3]. Among the compact surfaces, only the torus and the Klein bottle satisfy this condition.

We mention the following result without proof.

Theorem: If the topology of M (i.e., the system of open sets, cf. 5.3) is locally compact (which always holds for manifolds) and the second countability axiom is satisfied (there exists a countable basis for the topology), then there exists in every open covering an associated partition of unity, in the sense that $supp(f_i)$ is always contained in one of the given open sets.

For a proof, see for example [**40**]. In fact it is sufficient to make the (weaker) assumption that the space is paracompact.

[3] L. Markus, *Line element fields and Lorentz structures on differentiable manifolds*, Annals of Mathematics (2) **62**, 411–417 (1955).

Under the same assumptions there exists a Riemannian metric. In particular, the compactness of M implies the topological assumptions required. Thus, on an arbitrary compact manifold there exists a Riemannian metric.

5D The Riemannian connection

Just as at the beginning of Chapter 4, we have here the problem of defining the derivative on an abstract differentiable manifold or abstract Riemannian manifold not only for scalar functions (this is sufficiently done in the algebraic Definition 5.5), but also for vector fields. What we have to define is the notion of the derivative of a (tangent) vector field with respect to a tangent vector, with a result which is again a tangent vector. This will be defined in 5.13 in such a way that a Riemannian metric is not necessary and both arguments X and Y are treated equally. The so-called *Riemannian connection*, defined in 5.15, is nearer to the notion of covariant derivative of Chapter 4; in fact, it is just a generalization. Here we also require a compatibility with the Riemannian metric. The fundamental lemma of Riemannian geometry, presented in 5.16, shows the existence of a unique Riemannian connection for an arbitrary Riemannian metric.

5.13. Definition. (The Lie bracket[4])
Let X, Y be (differentiable) vector fields on M, and let $f\colon M \to \mathbb{R}$ be a differentiable function. Through the relation

$$[X,Y](f) := X(Y(f)) - Y(X(f))$$

we define a vector field $[X, Y]$, which is referred to as the *Lie bracket* of X, Y (also called the *Lie derivative* $\mathcal{L}_X Y$ of Y in the direction X). At a point $p \in M$ we have $[X,Y]_p(f) = X_p(Yf) - Y_p(Xf)$.

The Lie bracket measures the degree of non-commutativity of the derivatives. In Section 4.5 above we made a similar definition, namely $[X,Y] := D_X Y - D_Y X$, which in $\mathbb{R}^n$ is equivalent to the above definition. For the definition of the Lie bracket, no Riemannian metric is required; the differentiable structure is sufficient. The exercises

[4]Named after Sophus Lie, the founder of the theory of transformation groups.

at the end of the chapter help give a geometric interpretation and intuition of the Lie bracket. For scalar functions φ one simply sets $\mathcal{L}_X\varphi = X(\varphi)$ and declares in this way a Lie derivative for scalar functions and for vectors. There is also a Lie derivative in the direction of a vector field defined for one-forms given by the formula $\mathcal{L}_X\omega(Y) := X(\omega(Y)) - \omega(\mathcal{L}_X Y)$. On can similarly define a Lie derivative for tensor fields in general, see [**42**, 2.24]. If the Lie derivative vanishes in the direction of a vector field, this leads naturally to a corresponding notion of "constancy". An example is an isometric vector field X (also called a *Killing field*) on (M, g) characterized by the equation $\mathcal{L}_X g = 0$. Here we have $\mathcal{L}_X g(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$.

5.14. Lemma. (Properties of the Lie bracket)

Let X, Y, Z be vector fields, let α, β be real constants, and let $f, h\colon M \to \mathbb{R}$ be differentiable functions. Then the Lie bracket has the following properties:

(i) $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$;

(ii) $[X, Y] = -[Y, X]$;

(iii) $[fX, hY] = f \cdot h \cdot [X, Y] + f \cdot (Xh) \cdot Y - h \cdot (Yf) \cdot X$;

(iv) $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$; (*Jacobi identity*)

(v) $\left[\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j}\right] = 0$ for every chart with coordinates $(x^1, \ldots, x^n)$;

(vi) $\left[\sum_i \xi^i \dfrac{\partial}{\partial x^i}, \sum_j \eta^j \dfrac{\partial}{\partial x^j}\right] = \sum_{i,j} \left(\xi^i \dfrac{\partial \eta^j}{\partial x^i} - \eta^i \dfrac{\partial \xi^j}{\partial x^i}\right) \dfrac{\partial}{\partial x^j}$ (*representation in local coordinates*).

Proof: The properties (i) and (ii) are obvious. (iii) follows from the product rule 5.5:

$$[fX, hY](\phi) = fX((hY)\phi) - hY((fX)\phi)$$

$$= f(Xh)(Y\phi) + fhX(Y\phi) - h(Yf)(X\phi) - hfY(X\phi)$$

$$= \Big(fh[X, Y] + f(Xh)Y - h(Yf)X\Big)(\phi)$$

for every function ϕ in a neighborhood of the point under consideration.

(v) is nothing but the well-known Schwarz' law

$$\frac{\partial}{\partial x^i}\Big(\frac{\partial}{\partial x^j}(f)\Big) = \frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial}{\partial x^j}\Big(\frac{\partial}{\partial x^i}(f)\Big)$$

for the commutativity of the second derivatives.

The representation in (vi) has already occurred in Section 4.5, and is proved here in an entirely similar manner.

The Jacobi identity (iv) is easily checked as follows, where we symbolically write $[X, Y] = XY - YX$:

$$\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big]$$

$$= XYZ - XZY - YZX + ZYX + YZX - YXZ$$

$$-ZXY + XZY + ZXY - ZYX - XYZ + YXZ = 0.$$

□

5.15. Definition. (Riemannian connection)
A *Riemannian connection* ∇ (pronounced "nabla") on a Riemannian manifold (M, g) is a map

$$(X, Y) \longmapsto \nabla_X Y,$$

which associates to two given differentiable vector fields X, Y a third differentiable vector field $\nabla_X Y$, such that the following conditions are satisfied: ($f\colon M \to \mathbb{R}$ denotes a differentiable function):

(i) $\nabla_{X_1+X_2} Y = \nabla_{X_1} Y + \nabla_{X_2} Y$; (*additivity in the subscript*)

(ii) $\nabla_{fX} Y = f \cdot \nabla_X Y$; (*linearity in the subscript*)

(iii) $\nabla_X(Y_1 + Y_2) = \nabla_X Y_1 + \nabla_X Y_2$; (*additivity in the argument*)

(iv) $\nabla_X(fY) = f \cdot \nabla_X Y + (X(f)) \cdot Y$; (*product rule in the argument*)

(v) $X\big(g(Y, Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$; (*compatibility with the metric*)

(vi) $\nabla_X Y - \nabla_Y X - [X, Y] = 0.$ (*symmetry* or *torsion-freeness*)

REMARK: For simplicity one often uses the notation $\nabla_X f = X(f)$ for the directional derivative of f in the direction X. Dropping the conditions (v) and (vi) defines a plain "connection", and if the condition (vi) is not satisfied, the difference $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ is called the *torsion tensor* of ∇. Instead of "connection" one also speaks of a *covariant derivative* (cf. 4.3), and instead of "Riemannian connection", the term *Levi–Civita connection.*

The meaning of the term lies in a kind of "connection" between the different tangent spaces, which are disjoint by definition. This will occur again in sections 5.17 and 5.18, where the notion of parallel displacement (or parallel transport) of vectors in introduced. In this way it is possible to relate tangent vectors which are based at different points of the manifold. The properties for calculations with the Riemannian connection are identical to those of the covariant derivative in Section 4.4.

EXAMPLES:

1. In Euclidean space $(\mathbb{R}^n, g_o)$ with the standard metric g_0, we can set $\nabla = D$, which means that the directional derivative is a Riemannian connection, cf. the properties mentioned in Chapter 4.
2. On a hypersurface $M^n \to \mathbb{R}^{n+1}$, the covariant derivative in the sense of Definition 4.3 defines a Riemannian connection for the first fundamental form in the above sense.
3. In $\mathbb{R}^3$ set $\nabla_X Y := D_X Y + \frac{1}{2}(X \times Y)$, where $X \times Y$ is the usual cross product of vectors. This ∇ satisfies (i) - (v), but not (vi):

$$\nabla_X Y - \nabla_Y X = D_X Y - D_Y X + X \times Y = [X,Y] + \underbrace{X \times Y}_{\text{torsion}}$$

5.16. Theorem. On every Riemannian manifold (M,g) there is a uniquely determined Riemannian connection ∇.

PROOF: First we prove the *uniqueness.* From properties (i) - (vi) we get, for vector fields X, Y, Z, a relation as the sum of three equalities:

$$\left.\begin{array}{rcl} X\langle Y,Z\rangle &=& \langle\nabla_X Y,Z\rangle+\langle Y,\nabla_X Z\rangle \\ Y\langle X,Z\rangle &=& \langle\nabla_Y X,Z\rangle+\langle X,\nabla_Y Z\rangle \\ -Z\langle X,Y\rangle &=& -\langle\nabla_Z X,Y\rangle-\langle X,\nabla_Z Y\rangle \end{array}\right\}+$$

$$X\langle Y,Z\rangle+Y\langle X,Z\rangle-Z\langle X,Y\rangle=\langle Y,\underbrace{\nabla_X Z-\nabla_Z X}_{[X,Z]}\rangle+\langle X,\underbrace{\nabla_Y Z-\nabla_Z Y}_{[Y,Z]}\rangle$$
$$+\langle Z,\underbrace{\nabla_X Y+\nabla_Y X}_{2\nabla_X Y+[Y,X]}\rangle$$

From this we get the *Koszul formula*

$$(*)\qquad 2\langle\nabla_X Y,Z\rangle = X\langle Y,Z\rangle+Y\langle X,Z\rangle-Z\langle X,Y\rangle$$
$$-\langle Y,[X,Z]\rangle-\langle X,[Y,Z]\rangle-\langle Z,[Y,X]\rangle.$$

The right-hand side is uniquely determined, given Z; hence also $\nabla_X Y$ is uniquely determined.

To show the *existence* of ∇ we define ∇ by the requirement that $(*)$ holds for all X, Y, Z.

It remains to show that $(\nabla_X Y)\big|_p$ is defined (without using Z as a vector field), in other words, the expression $\langle\nabla_X Y\big|_p, Z_p\rangle$ depends only on Z_p, or equivalently,

$$\langle\nabla_X Y, f\cdot Z\rangle = f\cdot\langle\nabla_X Y,Z\rangle$$

for every scalar function f. This is easily verified by applying the properties of the Lie bracket and the product rule

$$X(fh)=f\cdot(Xh)+(Xf)\cdot h.$$

The validity of (i) - (vi) for the ∇ defined in this manner has to be established.

(i) and (iii) are obvious.

(ii) By applying the formula $(*)$ we get
$2\langle\nabla_{fX}Y,Z\rangle-2\langle f\nabla_X Y,Z\rangle$
$$=(Yf)\langle X,Z\rangle-(Zf)\langle X,Y\rangle-\langle Y,-(Zf)X\rangle-\langle Z,(Yf)X\rangle=0.$$

The proof of (iv) is similar.

(v) We have $2\langle\nabla_X Y, Z\rangle + 2\langle Y, \nabla_X Z\rangle$
$= X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle X,[Y,Z]\rangle - \langle Z,[Y,X]\rangle$
$+X\langle Z,Y\rangle + Z\langle X,Y\rangle - Y\langle X,Z\rangle - \langle Z,[X,Y]\rangle - \langle X,[Z,Y]\rangle - \langle Y,[Z,X]\rangle$
$= X\langle Y,Z\rangle + X\langle Z,Y\rangle = 2X\langle Y,Z\rangle$

(vi) We have $2\langle\nabla_X Y - \nabla_Y X, Z\rangle$
$= X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle - \langle Y,[X,Z]\rangle - \langle X,[Y,Z]\rangle - \langle Z,[Y,X]\rangle$
$-Y\langle X,Z\rangle - X\langle Y,Z\rangle + Z\langle Y,X\rangle + \langle X,[Y,Z]\rangle + \langle Y,[X,Z]\rangle + \langle Z,[X,Y]\rangle$
$= 2\langle [X,Y], Z\rangle.$ □

In *local coordinates* we get with the same formula the expression which we already met in Section 4.6 for the *Christoffel symbols*

$$\Gamma_{ij,k} = \frac{1}{2}\Big(- \frac{\partial}{\partial_k} g_{ij} + \frac{\partial}{\partial_j} g_{ik} + \frac{\partial}{\partial_i} g_{jk}\Big), \quad \Gamma_{ij}^m = \sum_k \Gamma_{ij,k} g^{km},$$

where $(g^{km}) := (g_{rs})^{-1}$, and

$$\Big\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \Big\rangle = \Gamma_{ij,k}, \quad \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

From this we get the following expression for $\nabla_X Y$, in local coordinates, provided $X = \sum_i \xi^i \frac{\partial}{\partial x^i}$ and $Y = \sum_j \eta^j \frac{\partial}{\partial x^j}$:

$$\nabla_X Y = \nabla_{\sum_i \xi^i \frac{\partial}{\partial x^i}} \Big(\sum_j \eta^j \frac{\partial}{\partial x^j}\Big)$$

$$= \sum_k \Big(\sum_i \xi^i \frac{\partial \eta^k}{\partial x^i} + \sum_{i,j} \Gamma_{ij}^k \xi^i \eta^j\Big) \frac{\partial}{\partial x^k}.$$

Especially for $X = \frac{\partial}{\partial x^i}$ we obtain

$$\nabla_X Y = \nabla_{\frac{\partial}{\partial x^i}}\Big(\sum_j \eta^j \frac{\partial}{\partial x^j}\Big) = \sum_k \Big(\frac{\partial \eta^k}{\partial x^i} + \sum_j \Gamma_{ij}^k \eta^j\Big) \frac{\partial}{\partial x^k}.$$

Consequently, in Ricci calculus the notation for this formula is

$$\nabla_i \eta^k = \frac{\partial \eta^k}{\partial x^i} + \Gamma^k_{ij}\eta^j.$$

In this expression, the left-hand side is not to be interpreted as the derivative of a scalar function η^k, but as the kth component of the derivative of the vector $(\eta^1, \ldots, \eta^n)$ with respect to the ith variable.

If we consider, instead of vector fields on the manifold itself, vector fields along a curve c, then the coordinate functions η^i are not to be viewed as functions of $x^1, \ldots, x^n$, but rather as functions of the curve parameter t. In this case, the following equation may be taken as a definition, where $c^1(t), \ldots, c^n(t)$ are the coordinates of c:

$$\nabla_{\dot{c}} Y = \sum_k \left(\frac{d\eta^k(t)}{dt} + \sum_{i,j} \dot{c}^i(t)\eta^j(t)\Gamma^k_{ij}(c(t)) \frac{\partial}{\partial x^k} \right)$$

$$= \sum_k \left(\sum_i \dot{c}^i(t) \frac{\partial \eta^k(t)}{\partial x^i} + \sum_{i,j} \dot{c}^i(t)\eta^j(t)\Gamma^k_{ij}(c(t)) \right) \frac{\partial}{\partial x^k}.$$

The Riemannian metric thus determines the Riemannian connection, and this in turn determines the notion of parallelness in the same way that the covariant derivative in the ambient Euclidean space did in Section 4.9.

5.17. Definition. (Parallel, geodesic, cf. also 4.9)

1. A vector field Y is said to be *parallel*, if $\nabla_X Y = 0$ for every X.
2. A vector field Y along a (regular) curve c is said to be *parallel along the curve* c, if $\nabla_{\dot{c}} Y = 0$ (this is independent of the parametrization).
3. A regular curve c is called a *geodesic*, if $\nabla_{\dot{c}}\dot{c} = \lambda\dot{c}$ for some scalar function λ. This is equivalent to the equation $\nabla_{c'}c' = 0$, provided c is parametrized by arc length.

The same remarks made in 4.9 for non-regular curves hold here also.

5.18. Corollary. (Parallel displacement, geodesics)

(i) Along an arbitrary regular curve c there is for each $Y_0 \in T_{c(t_0)}M$ a vector field Y (along c) which is parallel along c and whose value at $c(t_0)$ is Y_0. This vector field Y is called the *parallel displacement* of Y_0 along c.

(ii) Parallel displacement preserves the Riemannian metric, i.e., $\langle Y_1, Y_2\rangle$ is constant for any two parallel vector fields Y_1, Y_2 along c.

(iii) At every point p and for each $X \in T_pM$ with $g(X,X) = 1$, there is an $\epsilon > 0$ and a uniquely determined geodesic $c : (-\epsilon, \epsilon) \to M$ which is parametrized by arc length and for which $c(0) = p$, $\dot{c}(0) = X$.

The proof is literally the same as in 4.10, 4.11 and 4.12. It is sufficient to consider the parts of the curve which are contained in local charts. The equation which $Y(t) = \sum_j \eta^j(t)\frac{\partial}{\partial x^j}$ satisfies if and only if it is parallel along c: $\nabla_{\dot{c}}Y = 0$ (where $x^i(t)$ are the coordinates of c) is

$$\frac{d\eta^k}{dt} + \sum_{i,j} \dot{x}^i(t) \cdot \eta^j(t) \cdot \Gamma^k_{ij}(c(t)) = 0, \quad k = 1, \dots, n.$$

The system of equations which c satisfies precisely if it is a geodesic is

$$\frac{d^2x^k}{dt^2} + \sum_{i,j} \dot{x}^i(t) \cdot \dot{x}^j(t) \cdot \Gamma^k_{ij}(c(t)) = 0, \quad k = 1, \dots, n.$$

5.19. Definition. (Exponential mapping)
For a fixed point $p \in M$ let $c_V^{(p)}$ denote the uniquely determined geodesic through p which is parametrized by arc length in the direction of a unit vector V. In some neighborhood U of $0 \in T_pM$, the following mapping is well-defined:

$$T_pM \supseteq U \ni (p, tV) \longmapsto c_V^{(p)}(t).$$

Here the parameters are chosen is such a way that $(p, 0) \mapsto p$. This map is called the *exponential mapping* at the point p, and it is denoted by $\exp_p \colon U \longrightarrow M$. For variable points p one can define a mapping exp in a similar manner by the formula $\exp(q, tV) = \exp_q(tV) = c_V^{(q)}(t)$. This can be defined on an open set of the tangent bundle TM, compare exercise 3.

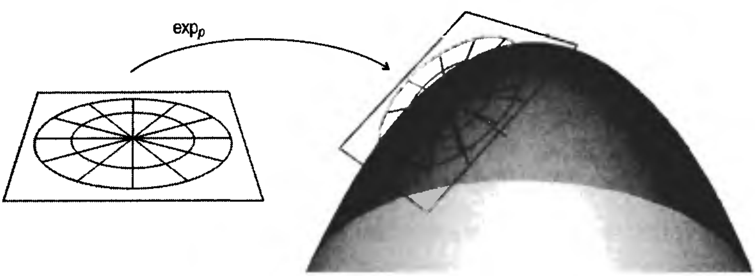

Figure 5.2. Exponential mapping at a point p

REMARK: $\exp_p$ maps the lines through the origin of T_pM to geodesics, and this mapping is isometric because the arc length is preserved, see Figure 5.2. In all directions perpendicular to the geodesics through p the map $\exp_p$ is in general not isometric, i.e., it is not length-preserving. This question will be addressed again later in Section 7B, where a more precise study of the transformation of lengths is made.

EXAMPLES:

1. In $\mathbb{R}^n$ the exponential mapping is nothing but the canonical identification of the tangent space $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ itself, where the origin of the tangent space is mapped to the point p. More precisely, $\exp_p(tV) = p + tV$.

2. For the unit sphere S^2 with south pole $p = (0, 0, -1)$, the exponential mapping can be expressed in the following manner using polar coordinates, where we write a tangent vector as $r\cos\phi\frac{\partial}{\partial x} + r\sin\phi\frac{\partial}{\partial y}$, thus viewing it as a function of r and ϕ:

$$\exp_p(r, \phi) = \Big(\cos\phi\cos\big(r - \frac{\pi}{2}\big), \sin\phi\cos\big(r - \frac{\pi}{2}\big), \sin\big(r - \frac{\pi}{2}\big)\Big).$$

The circle $r = \frac{\pi}{2}$ in the tangent plane gets mapped to the equator, while the circle $r = \pi$ maps to the north pole. At this point the exponential mapping degenerates.

3. In the group $\mathbf{SO}(n, \mathbb{R})$ with the unit element E and with the (bi-invariant) standard metric, $\exp_E$ is given by the exponential series

$$A \longmapsto \sum_{k\geq 0} \frac{A^k}{k!}$$

evaluated for an arbitrary skew-symmetric real $(n \times n)$-matrix A (cf. the proof of 2.15). This is the origin of the name *exponential mapping.* The exponential rule

$$\exp\big((t + s)A\big) = \exp(tA) \cdot \exp(sA)$$

expresses the fact that the line $\{tA \mid t \in \mathbb{R}\}$ is mapped by $\exp_E$ onto a 1-parameter subgroup of matrices. A very similar state of affairs holds for other matrix groups such as $\mathbf{GL}(n, \mathbb{R}), \mathbf{SL}(n, \mathbb{R})$, $\mathbf{U}(n), \mathbf{SU}(n)$. This mapping is of fundamental importance in the theory of Lie groups. The tangent space at the unit element is the corresponding Lie algebra. In the case of the rotation group $\mathbf{SO}(n, \mathbb{R})$, the Lie algebra is the set of skew-symmetric $(n \times n)$-matrices, together with the multiplication given by the commutator $[X, Y] = XY - YX$. (Compare with 5.13.) For more details see [**43**], Chapter 1.

5.20. Definition. (Holonomy group)

Let $P^c : T_pM \longrightarrow T_pM$ denote the parallel translation along a closed curve c with $c(0) = c(1) = p$. For this it suffices that c is continuous and piecewise regular, since the parallel translation is the composition of the corresponding smooth parts and one may then apply 5.18 (i).

For c_1 and c_2 let $c_2 * c_1$ denote the composition of the curves, and let $c^{-1}(t) := c(L-t)$ for $c\colon [0,L] \to M$ (run through in the opposite direction). Then one has

$$P^{c_2 * c_1} = P^{c_2} \circ P^{c_1},$$
$$P^{c^{-1}} = \left(P^c\right)^{-1},$$

and the set of all parallel translations from p to p along piecewise regular curves thus has the structure of a group. It is called the *holonomy group* of the manifold (M,g) at the point p. If M is path connected, then all holonomy groups are isomorphic to each other and one just speaks of *the* holonomy group of (M,g). The holonomy group is always a subgroup of the orthogonal group $\mathbf{O}(n)$, which operates on $T_pM \cong \mathbb{R}^n$. This follows from 5.18 (ii).

EXAMPLES:

1. The holonomy group is trivial for $\mathbb{R}^n$ and for the flat torus $\mathbb{R}^n/\mathbb{Z}^n$. The reason for this is that the parallel translation in the sense of the Riemannian metric coincides with the usual parallel translation. For every closed path the result under parallel translation is the vector one starts with.
2. On the standard sphere S^2 the holonomy group contains all rotations (exercise).
3. On a flat cone with a non-trivial opening angle (this is a ruled surface with $K = 0$, cf. 3.24), the holonomy group is not trivial. This is seen by cutting the cone open and developing it on the plane (cf. 3.24). The identification at the boundary leads to non-trivial elements of the holonomy group.
4. On a flat Möbius strip the holonomy group also contains a reflection. This can again be seen most easily by developing the surface in the plane.

Exercises

1. Show that the open hemispheres $\{(x_1,x_2,x_3) \in S^2 \mid x_i \neq 0\}$ for $i = 1,2,3$ define an atlas of the two-dimensional sphere with six (connected) charts $U_1,\ldots,U_6$. Here x_1,x_2,x_3 denote Cartesian coordinates. Determine explicitly the transformation functions

between the charts. A picture of the six hemispheres can be found on the cover of the book [**14**].

2. Show that the Cartesian product $M_1 \times M_2$ of two differentiable manifolds is again a differentiable manifold.

3. Show that for a given differentiable manifold M the set of all pairs (p, X) with $X \in T_pM$ is again (in a natural way) a differentiable manifold; it is called the *tangent bundle* TM of M. For this, construct for every chart φ in M an *associated bundle chart* by means of

$$\Phi(p, X) := (\varphi(p), \xi^1(p), \cdots, \xi^n(p)) \in I\!R^n \times I\!R^n,$$

where $\xi^1, \ldots, \xi^n$ are the components of X in the corresponding basis, i.e., $X_p = \sum_{i=1}^n \xi^i(p) \frac{\partial}{\partial x^i}\big|_p$. Check the properties of Definition 5.1. (Note: Formally the definition of the tangent bundle includes the projection from TM to M given by $(p, X) \mapsto p$.)

4. Determine whether this definition of the tangent bundle coincides in the case of $M = I\!R^n$ with Definition 1.6.

5. Show the following. The tangent bundle of the unit circle S^1 is diffeomorphic to the cylinder $S^1 \times I\!R$. The analogous statement does not hold for the two-sphere S^2, but surprisingly it *does* hold for the three-sphere S^3: the tangent bundle of S^3 is diffeomorphic to the product $S^3 \times I\!R^3$, cf. the exercises at the end of Chapter 7.

6. The metrics on two Riemannian manifolds (M_1, g_1) and (M_2, g_2) induce in a canonical manner a Riemannian metric $g_1 \times g_2$ on the Cartesian product $M_1 \times M_2$, the so-called *product metric*. What is the form of this metric in local coordinates?

7. Let a submanifold M of $I\!R^4$ be given by the equation

$$M = \{(x_1, x_2, x_3, x_4) \in I\!R^4 \mid x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1\}.$$

Prove that M is a two-dimensional manifold by displaying an explicit atlas.

8. Construct an explicit Lorentzian metric, i.e., a metric tensor of type $(-+)$, on the (abstract) Klein bottle (cf. the examples following 5.1 in the text).

9. Let (M, g) be a two-dimensional Riemannian manifold, and let $\Delta \subset M$ be a geodesic triangle which is the boundary of a simply connected domain. Show that the parallel translation along this boundary (traced through once) is a rotation in the tangent plane. Calculate the angle of rotation in terms of quantities which only depend on the interior of Δ. Hint: Gauss-Bonnet formula.

10. Show that the holonomy group of the standard two-sphere S^2 really contains all the rotations. Hint: Consider curves which are constructed piecewise from great circles.

11. Determine the holonomy group of the hyperbolic plane as a surface in three-dimensional Minkowski space (cf. 3.44). Here the covariant derivative is to be taken as in Euclidean space, that is, with tangent components which are directional derivatives.

12. Let (M_*, g_*) be an n-dimensional Riemannian manifold and let $f\colon \mathbb{R} \to \mathbb{R}$ be a function without zeros. Then $\mathbb{R} \times M$ endowed with the metric

$$g(t, x^1, \ldots, x^n) = dt^2 + (f(t))^2 \cdot g_*(x^1, \ldots, x^n)$$

is again a Riemannian manifold, the so-called *warped product* with the *warping function* f. Show that the t-lines are always geodesics. What are sufficient conditions in order that geodesics on M_* are also geodesics on M?

13. Let X be a vector field on the manifold M. Show the following.
    (a) At every point $p \in M$ there is a uniquely determined curve $c_p\colon I_p \to M$ with $c_p(0) = p, c_p'(t) = X_{c(t)}$, where I_p is the maximal interval around $t = 0$ with this property.
    (b) For every open neighborhood U of p there is a set, open in $\mathbb{R} \times M$, such that the map ψ which is defined by $\psi(t, q) := \psi_t(q) := c_q(t)$ is differentiable. ψ is called the *local flow* of X at the point p.
    (c) In case ψ_t is defined for every $t \in \mathbb{R}$, one calls the vector field (or also the flow) *complete*. In this case one has $\psi_{t+s} = \psi_t \circ \psi_s$ for all $t, s \in \mathbb{R}$. This property defines a one-parameter group of diffeomorphisms, since $t \mapsto \psi_t$ is a group homomorphism. Why are all ψ_t diffeomorphisms?

14. Let X be a vector field on an n-dimensional manifold M with $X_p \neq 0$ at a point $p \in M$. Using the previous exercise, show that there is a coordinate system $x^1, \ldots, x^n$ near p with $X = \frac{\partial}{\partial x^1}$.
15. Let X, Y be vector fields on M, and let ψ denote the local flow of X at a point $p \in M$. Again using the previous exercise, verify the following equation:
$$[X, Y] = \lim_{t \to 0} \frac{1}{t}\Big(D\psi_{-t}(Y_{\psi_t(p)}) - Y_p\Big).$$
16. Show that the tangent space of the rotation group $\mathbf{SO}(3)$ at the "point" corresponding to the identity matrix can be identified in a natural manner with the set of all skew-symmetric (3×3)-matrices (cf. also the proof of 2.15). Calculate the differential of the Cayley map $CAY\colon \mathbb{R}^3 \to \mathbf{SO}(3)$. For the definition of this map see the examples following 5.1.
17. Give an explicit atlas for the manifold $\mathbb{R}P^3$ (*real projective space*), which is defined as the quotient of the three-sphere by the antipodal mapping.
18. Show that the exponential series
$$A \longmapsto \sum_{k \geq 0} \frac{A^k}{k!}$$
is actually an orthogonal matrix for an arbitrary skew-symmetric matrix A.
19. Find a formula for the inverse mapping of the exponential mapping (a kind of *logarithm*) for the case of the group $\mathbf{SO}(n)$. Hint: Take a power series and determine the coefficients.
20. The *Schwarzschild half-plane* is defined as the half-plane $E = \{(t, r) \in \mathbb{R}^2 \mid r > r_0\}$ with the semi-Riemannian metric $ds^2 = -hdt^2 + h^{-1}dr^2$, where h denotes the function $h(r, t) := 1 - r_0/r$. Show that the maps $(t, r) \mapsto (\pm t + b, r)$ are isometries. Moreover, calculate the Christoffel symbols and show that the r-lines are always geodesics. Show also that for the geodesics, written $\gamma(s) = (t(s), r(s))$, the quantity $h(\gamma(s))t'(s)$ is a constant. The constant r_0 corresponds to the Schwarzschild radius, which depends on the mass of a black hole, which one should imagine is situated at $r = 0$.

21. Suppose we are given coordinates in (M, g) such that in these coordinates the metric tensor has diagonal form, i.e., $g_{ij} = 0$ for $i \neq j$. Show that the system of equations for geodesics is as follows:

$$\frac{d}{ds}\left(g_{kk}\frac{dx^k}{ds}\right) = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial g_{ii}}{\partial x^k}\left(\frac{dx^i}{ds}\right)^2 \qquad (k = 1, \ldots, n).$$

22. Let the *Schwarzschild metric* be given as follows:

$$ds^2 = -h \cdot dt^2 + h^{-1} \cdot dr^2 + r^2\left(\sin^2 \vartheta d\varphi^2 + d\vartheta^2\right),$$

where $h = h(r) = 1 - \frac{2M}{r}$. The Schwarzschild metric is a model for a universe in which there is precisely one rotationally symmetric star. Show that every geodesic c satisfies the following equations with constants E and L:
(a) $h \cdot \frac{dt}{ds} = E$,
(b) $r^2 \sin^2 \vartheta \cdot \frac{d\varphi}{ds} = L$,
(c) $\frac{d}{ds}\left(r^2 \cdot \frac{d\vartheta}{ds}\right) = r^2 \sin\vartheta \cos\vartheta \left(\frac{d\varphi}{ds}\right)^2$.
Now suppose that c is parametrized by arc length τ, which describes a freely falling particle (in particular, this implies it is not a light particle, for which $g(c', c') \neq 0$ holds), with the initial condition that it is falling equatorially, i.e., satisfies $\vartheta(0) = \frac{\pi}{2}$ and $\frac{d\vartheta}{ds}(0) = 0$. Then we have
(a′) $h \cdot \frac{dt}{d\tau} = E$,
(b′) $r^2 \frac{d\varphi}{d\tau} = L$,
(c′) $\vartheta = \frac{\pi}{2}$.
Hint: Exercise 21.

Chapter 6

The Curvature Tensor

In the Gauss equation 4.15 or 4.18, we have on the left-hand side an expression which we called the curvature tensor. Its connection to the curvature (and thus the nomenclature) is clarified by the *Theorema Egregium* 4.16 and 4.20. For this it is of great importance that the left-hand side of the equation only depends on the first fundamental form or the covariant derivative, which follows from the equation

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

in the Koszul-style calculus, or

$$R^s_{ikj} = \frac{\partial \Gamma^s_{ij}}{\partial u^k} - \frac{\partial \Gamma^s_{ik}}{\partial u^j} + \Gamma^r_{ij}\Gamma^s_{rk} - \Gamma^r_{ik}\Gamma^s_{rj}$$

in Ricci calculus. (The more precise notation here would be $R^{s}_{\cdot ikj}$ instead of R^s_{ikj}.) This expression is well-defined for an arbitrary Riemannian manifold and is the foundation for all further information on curvature of Riemannian manifolds. In fact, all scalar curvature quantities can be obtained from this curvature tensor. Before we go into this, we make a brief digression with some general remarks on tensors.

6A Tensors

Tensors are operators which are not determined by the process of taking derivatives of other quantities (a local process), but rather through evaluation of known quantities at single points. An example